($\lambda_B \gg 1$) could significantly influence the aerodynamic properties of a massively-ablating planetary entry vehicle. Further parametric studies and extension of the present approach to turbulent flow are in progress, including consideration of upstream influence effects on finite length bodies and comparisons with available exact numerical solutions for massive blowing.

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Cross-Flow Effects on the Boundary Layer in a Plane of Symmetry

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Nomenclature

A = constant in skin friction law (see Eq. 12); $A = 0.0128$ for turbulent flow, $A = 0.22$ for laminar flow
C_f = local skin friction coefficient $= \tau_w/\rho_e u_e^2$
K_θ = correction factor on momentum thickness due to cross flow (Eq. 18)
K_{c_f} = correction factor due to cross flow (Eq. 17)
I, J, K = unit vectors in cartesian coordinates X, Y, Z (Fig. 1)
$\hat{i}, \hat{j}, \hat{k}$ = unit vectors in streamwise coordinates (Fig. 1)
e_v = unit vector in freestream direction (Fig. 1)
$\hat{m}$ = unit vector normal to both e_v and $\hat{n}$ (Fig. 1)
$\hat{n}$ = unit vector normal to surface (Fig. 1)
M = Mach number
n = exponent in skin friction law (Eq. 12), $n = 1.0$ for laminar flow, $n = 0.25$ for turbulent flow
p = pressure
p_2 $= \frac{1}{2}\partial^2 p/\partial\phi^2$ (Eq. 5)
$\dot{q}_w$ = heat-transfer rate at the wall
R = transverse radius of curvature of the body (Fig. 1)
R_{e_θ} = Reynolds number based on momentum thickness $= \rho_e u_e \Theta/\mu_e$
u_e = streamwise external velocity along x
v = cross-flow velocity derivative $= \partial w/\partial\phi$
V = total velocity
w = cross-flow velocity
x, y, z = Cartesian coordinates in streamline direction (Fig. 1)
X, Y, Z = Cartesian coordinates in body axes (Fig. 1)
α = angle of attack
δ = boundary layer thickness
δ^* = displacement thickness
θ = angle between the freestream velocity vector and normal to the surface (Fig. 1)
θ_c' = angle that the generatrix of the conical body makes with its axis
θ_0 = value of the angle θ in the plane of symmetry ($\phi = 0$)
Θ = momentum thickness
μ = viscosity
ϕ = meridional angle (Fig. 1)
ρ = density
τ_w = shear stress at the wall
ω = cross-flow parameter (Eq. 2)

Subscripts

e = edge condition
∞ = freestream condition
0 = value at plane of symmetry, $\phi = 0$
x = derivative with respect to x
X = derivative with respect to X
ϕ = derivative with respect to ϕ
$*$ = reference enthalpy condition

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AN engineering method for estimating the boundary-layer properties on bodies in hypersonic flow is the "effective cone analysis." In this analysis, the flow properties external to the boundary layer are assumed to be the same as those existing on an equivalent cone of half angle equal to the cone half angle plus the angle of attack. This technique has been found to be in error,[1] especially in laminar boundary-layer regions.

In the present Note, simple engineering formulas, valid for noncircular cones at angle of attack, are derived to show that the effective cone analysis accounts only for the increase in the local edge Reynolds number due to the angle of attack. A correction factor (which depends on the cross flow produced by the transverse curvature of the body and on the angle of attack) to the effective cone analysis gives good agreement with exact theory. The present analysis is valid for hypersonic flight speeds and small cross-flow approximations.

The integral equations are analogous to those presented in Refs. 2–4. For conical bodies, the streamwise gradients vanish and the streamwise integral momentum equation reduces to (see Eq. 1 of Ref. 4).

$$\frac{d\Theta}{dx} + \Theta\frac{d\ln R}{dx} + \frac{v_e}{Ru_e}\int_0^\delta \frac{\rho}{\rho_e}\frac{v}{v_e}\left(1 - \frac{u}{u_e}\right)dz = \frac{\tau_w}{\rho_e u_e^2} = \frac{C_f}{2} \tag{1}$$

Note that in Ref. 4 the term v/v_e is assumed constant.

For conical bodies, $dv_e/dx = 0$ and v_e is obtained from (see Eq. 3 of Ref. 4)

$$\omega = v_e/R_x u_e = -\tfrac{1}{2}\{1 - [1 - 8p_2/(\rho_e u_e^2 R_x^2)]^{1/2}\} \tag{2}$$

For small cross flow

$$v_e/u_e \simeq -2p_2/\rho_e u_e^2 R_x \tag{3}$$

Where

$$p_2 = \tfrac{1}{2}\partial^2 p/\partial\phi^2 \tag{4}$$

is calculated from Newtonian theory,

$$p = p_\infty + \rho_\infty V_\infty^2 \sin^2\theta \simeq p_0 + p_2\phi^2 \quad \text{for} \quad \phi \ll 1 \tag{5}$$

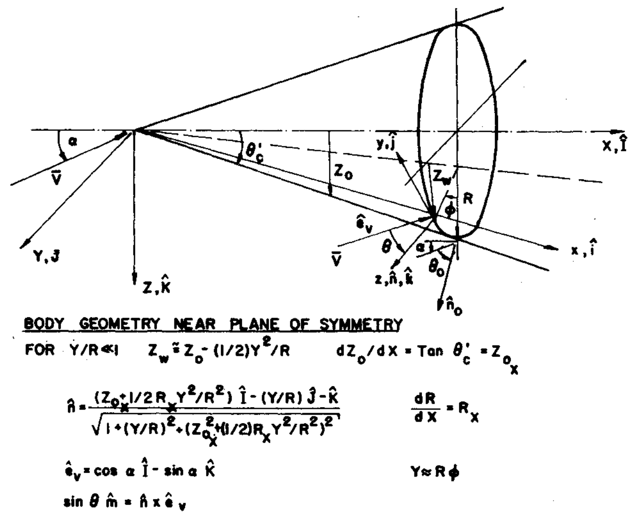

Fig. 1 Coordinate systems and body geometry.

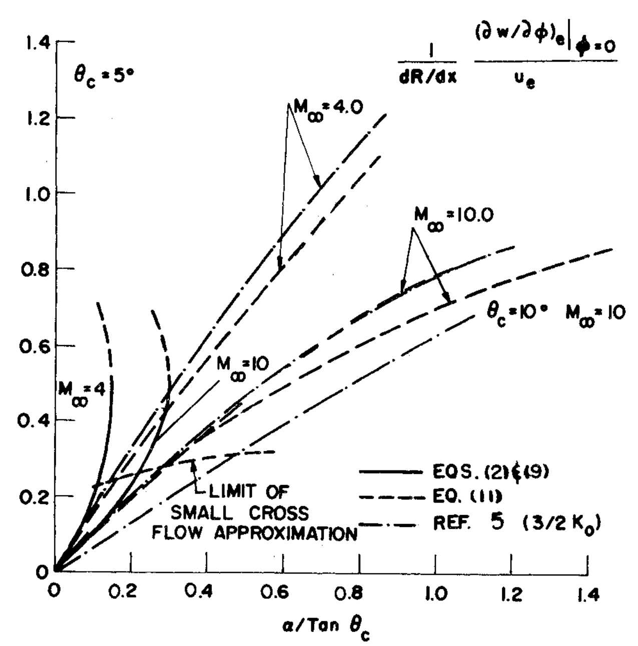

Fig. 2 Cross-flow velocity parameter.

Near the plane of symmetry (Fig. 1), the angle θ is given by

$$\sin^2\theta \simeq \sin^2\theta_0[1+\theta_2\phi^2] \tag{6a}$$

where

$$\theta_2 \simeq \left[\left(\frac{\cos^2\theta_c'}{\cos^2(\theta_c'+\alpha)}-1\right)-\sin\alpha\left(\frac{\cos\theta_c'}{\cos(\theta_c'+\alpha)}\right)R_X + \sin^2\theta_c'\left(1-\frac{R_X}{\tan\theta_c'}\right)\right] \tag{6b}$$

and

$$\theta_0 = \pi/2-(\theta_c'+\alpha) \tag{7}$$

Hence

$$p_0 = p_\infty + \rho_\infty V_\infty^2\cos^2(\theta_c'+\alpha) \tag{8}$$

and

$$p_2 = (\rho_\infty V_\infty^2\sin^2\theta_0)\theta_2 \tag{9}$$

The first term in the brackets in Eq. (6b) is solely due to angle of attack, while the second term is due to either angle of attack or transverse curvature. The third term is due only to transverse curvature.

For conical bodies, the transverse radius of curvature of the body is linearly proportional to the streamwise direction x, i.e., $R = \pm Cx$. The double sign is to include convex as well as concave conical bodies. For example, for elliptic cones, $R = (a/b)a$ or $R = (b/a)b$, depending on whether the minor or major axis is in the plane of symmetry; while for circular cones,

$$R = x\sin\theta_c' = X\tan\theta_c' \tag{10}$$

(Fig. 1), so that $X = x\cos\theta_c'$. Therefore, for conical bodies the inviscid cross-flow derivative is constant and is given by combining Eqs. (2) and (9). For small cross flow, this becomes

$$\omega \simeq -2\left(\frac{\rho_\infty V_\infty^2}{\rho_e u_e^2}\right)\left(\frac{\cos(\theta_c'+\alpha)}{\sin\theta_c'}\right)^2 \tag{11}$$

Equation 11 shows that the cross-flow velocity derivative at a plane of symmetry is proportional to the ratio of the freestream momentum to the momentum at the edge of the viscous layer. This function is shown plotted in Fig. 2 as a function of flight Mach number and angle of attack for a 5° cone. The exact values given by Jones[5] are also shown for comparison. The small cross-flow approximations gives smaller values of ω at large angles of attack than those reported in Ref. 5.

Assuming the cross-flow velocity gradient profile is similar to the streamwise velocity profile, i.e., $v/v_e = u/u_e$, and the skin friction coefficient is given by a modified Prandtl skin friction law

$$\tfrac{1}{2}C_f = (A/R_{e_\theta}^{\,n})(\mu^*/\mu_e)^n(\rho^*/\rho_e) \tag{12}$$

The integral momentum equation reduces to

$$\frac{d\Theta^{n+1}}{dx} + (n+1)\,\Theta^{n+1}\left[\frac{d\ln R}{dx}+\frac{v_e}{Ru_e}\right] = A(n+1)\left(\frac{\rho_e u_e}{\mu_e}\right)^{-n}\left(\frac{\mu^*}{\mu_e}\right)^n\left(\frac{\rho^*}{\rho_e}\right) \tag{13}$$

The solution of which, subject to the initial condition $\Theta = 0$ at $x = 0$, is

$$\Theta = \left[\left(\frac{A(n+1)}{(n+2)\pm(n+1)\omega}\right)\left(\frac{\rho^*}{\rho_e}\right)\left(\frac{\mu^*}{\mu_e}\right)^n\left(\frac{\rho_e u_e}{\mu_e}\right)^{-n}\right]^{1/(n+1)} x^{1/(n+1)} \tag{14}$$

Similar relations can be derived for the displacement thickness, boundary layer thickness, and heat-transfer rate if one assumes a Crocco-type relation for the temperature profile. Equation (14) shows that the rate of boundary layer growth (i.e., exponent of x) is independent of the transverse curvature and angle of attack. Consider the parameter ω in Eq. (14). For conical bodies at small angles of attack this reduces [see Eq. (11)] to

$$\omega \simeq -2\left(\frac{\rho_\infty V_\infty^2}{\rho_e u_e^2}\right)\cos^2\theta_c'\left(\frac{\sin\theta_c'}{R_X}\right)^2\left[\left(1+\frac{\alpha}{\tan\theta_c'}\right)\left(1-\frac{R_X}{\tan\theta_c'}\right)+\frac{\alpha}{\sin\theta_c'\cos\theta_c'}\right] \tag{15}$$

Further, for slender cones $\alpha \ll \theta_c' \ll 1$, provided $\delta \ll R$,

$$\omega \simeq -2\left(\frac{\rho_\infty V_\infty^2}{\rho_e u_e^2}\right)\left(\frac{\theta_c'}{R_X}\right)^2\left[1+\frac{2\alpha}{\theta_c'}\mp\frac{R_X}{\theta_c'}\right] \tag{16}$$

Therefore the transverse curvature is important for bodies whose radius of curvature is small compared to the streamwise slope of the conical body in the plane of symmetry. For circular cones ($R_X = \tan\theta_c' \simeq \theta_c'$) the effects are due solely to angle of attack through the cross-flow term and the different local edge Reynolds number.

The correction factor, due solely to cross-flow effects that must be applied to the effective cone analysis for the skin-friction coefficient is

$$K_{c_f} = \frac{C_f}{C_{f_{v_e=0}}} = \left[1\pm\left(\frac{n+1}{n+2}\right)\omega\right]^{-n/(n+1)} \simeq 1\pm\left(\frac{n}{n+2}\right)\omega \tag{17}$$

The plus sign is for outflow (i.e., convex bodies in windward plane). The minus sign is for inflow (i.e., concave bodies in windward plane or convex bodies in leeward plane).

For the case reported in Ref. 8 ($\theta_c = 6°$, $\alpha = 4°$, $M_\infty = 10.17$), K_{c_f} is equal to 1.27 in the laminar region, and 1.09 in the turbulent region. These correction factors account for the discrepancies noted in Ref. 8 between the values of skin friction predicted by the effective cone analysis and exact theory. The correction factor for the momentum thickness is

$$K_\theta = \frac{\Theta}{\Theta_{v_e=0}} = \left[1\pm\frac{n+1}{n+2}\omega\right]^{1/(n+1)} \simeq 1\mp\left(\frac{1}{n+2}\right)\omega \tag{18}$$

Hence, the effective cone analysis overpredicts the momentum thickness in both laminar and turbulent regions. The factor $(n/n+2)$ demonstrates the reason the skin friction is affected more by cross flow in laminar regions (where $n/n+2 = \frac{1}{3}$) than in turbulent regions (where $n/n+2 = \frac{1}{9}$); whereas, the effects on the momentum thickness is of the same order in either region since ($1/n+2 = \frac{1}{3}$) in the laminar region and ($1/n+2 = \frac{4}{9}$) in turbulent regions.

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